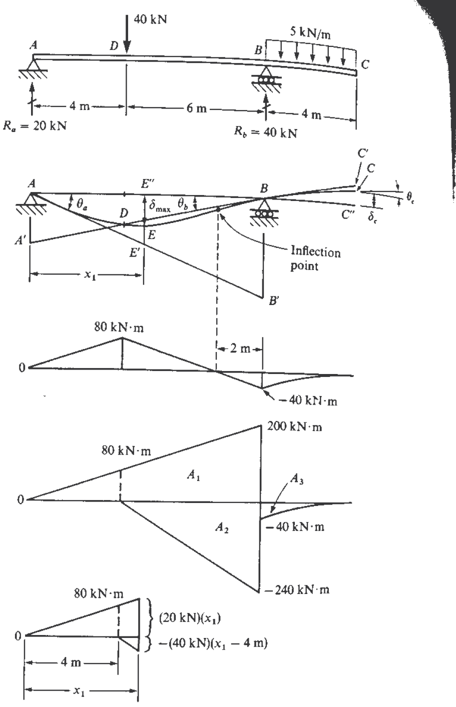


Example 5. Beam with an

that the angles are positive as shown in the figure. (b) Calculate the deflection δ_c at the free end C. (c) Calculate the maximum deflection δ_{max} in span AB.

(a) To find the various angles of rotation and deflections, we need the bending-moment diagram. From static equilibrium, we construct the diagram shown below the sketch of the deflection curve. The bending moment under the concentrated load is 80 kN·m and at support B is −40 kN·m. Between these points the bending moment varies linearly and becomes zero at a point located 2 m from B. At the point of zero moment, the bending moment changes sign; hence, the curvature also changes sign. Consequently, there is a point of zero curvature, called an **inflection point**, or **point of contraflexure**, in the deflection curve of the beam. To the left of the inflection point, the beam bends concave upward; to the right, it bends concave downward.

For convenience in calculating areas and first moments of the M/EI diagram, we shall redraw the bending-moment diagram as shown in the fourth sketch in the figure. This moment diagram is equivalent to the one just above it, as can easily be verified by calculating the bending moment at a few selected points. The upper triangle in this sketch represents the moment from A to B of the reaction at A, and the lower triangle represents the moment of the concentrated load from D to B. A bending-moment diagram constructed in this form is referred to as a moment diagram drawn by "parts," for the obvious reason that, instead of giving the total bending moment at any cross section, the diagram gives the moment in parts. We can use either form of the bending-moment diagram when making calculations by the moment-area theorems, but in this example it is easier to use the diagram drawn by parts. Of course, the total bending moment is needed when designing the beam.

As a preliminary matter, let us calculate the areas A_1, A_2, and A_3 of the three parts of the bending-moment diagram:

$$A_1 = \frac{1}{2}(10\ \text{m})(200\ \text{kN·m}) = 1000\ \text{kN·m}^2$$

$$A_2 = \frac{1}{2}(6\ \text{m})(-240\ \text{kN·m}) = -720\ \text{kN·m}^2$$

$$A_3 = \frac{1}{3}(4\ \text{m})(-40\ \text{kN·m}) = -53.33\ \text{kN·m}^2$$

The corresponding areas of the M/EI diagram are obtained by dividing these areas by EI.

Now we are ready to calculate the angle of rotation θ_a (Fig. 7-13). This angle equals the distance BB' divided by the span length of 10 m. The distance BB' equals the first moment of the area of the M/EI diagram between A and B, taken about B. Therefore, the quantity EI times the distance BB' is calculated as follows:

$$EI(BB') = A_1\left(\frac{10\ \text{m}}{3}\right) + A_2\left(\frac{6\ \text{m}}{3}\right)$$

$$= (1000\ \text{kN·m}^2)\left(\frac{10\ \text{m}}{3}\right) - (720\ \text{kN·m}^2)\left(\frac{6\ \text{m}}{3}\right)$$

$$= 1893\ \text{kN·m}^3$$

The quantity $EI\theta_a$ can now be calculated:

$$EI\theta_a = \frac{EI(BB')}{10\ \text{m}} = 189.3\ \text{kN·m}^2$$

Note that, for convenience in the calculations, we keep EI as a common factor. Later, we will substitute numerical values for E and I and determine the value of θ_a in radians.

The angle of rotation θ_b is determined in a similar manner. We first find the distance AA' from the second moment-area theorem:

$$EI(AA') = A_1\left(\frac{2}{3}\right)(10\ \text{m}) + A_2\left[4\ \text{m} + \frac{2}{3}(6\ \text{m})\right]$$

$$= (1000\ \text{kN·m}^2)\left(\frac{20\ \text{m}}{3}\right) - (720\ \text{kN·m}^2)(8\ \text{m})$$

$$= 906.7\ \text{kN·m}^3$$

Hence, the angle θ_b (times EI) is

$$EI\theta_b = \frac{EI(AA')}{10\ \text{m}} = 90.67\ \text{kN·m}^2$$

The angle of rotation θ_c is equal to the angle θ_b at support B plus the area of the M/EI diagram between B and C, in accord with the first moment-area theorem. Hence,

$$EI\theta_c = EI\theta_b + A_3$$

$$= 90.67\ \text{kN·m}^2 - 53.33\ \text{kN·m}^2 = 37.33\ \text{kN·m}^2$$

Now we can determine the actual angles of rotation by substituting E = 200 GPa and $I = 1.28 \times 10^9\ \text{mm}^4$ into the preceding equations. The product EI equals 256.0 MN·m²; therefore,

$$\theta_a = \frac{189.3\ \text{kN·m}^2}{256.0\ \text{MN·m}^2} = 739 \times 10^{-6}\ \text{rad}$$

$$\theta_b = \frac{90.67\ \text{kN·m}^2}{256.0\ \text{MN·m}^2} = 354 \times 10^{-6}\ \text{rad}$$

$$\theta_c = \frac{37.33\ \text{kN·m}^2}{256.0\ \text{MN·m}^2} = 146 \times 10^{-6}\ \text{rad}$$

Thus, the required angles of rotation have been calculated.

(b) From Fig. 7-13, we see that the deflection δ_c equals the distance CC'' minus the distance $C'C$. The first of these distances is obtained by multiplying θ_b by the distance from B to C:

$$EI(C'C'') = EI\theta_b(4\ \text{m}) = (90.67\ \text{kN·m}^2)(4\ \text{m})$$

$$= 362.7\ \text{kN·m}^3$$

The distance $C'C$ is the offset of point C from the tangent at B, which is equal to the negative of the first moment of the area of the M/EI diagram between B and C, with respect to C:

$$EI(C'C) = -A_3\left(\frac{3}{4}\right)(4\ \text{m}) = (53.33\ \text{kN·m}^2)(3\ \text{m})$$

$$= 160.0\ \text{kN·m}^3$$

Therefore, the deflection (times EI) is

$$EI\delta_c = EI(C'C'') - EI(C'C)$$

$$= 362.7\ \text{kN·m}^3 - 160.0\ \text{kN·m}^3 = 202.7\ \text{kN·m}^3$$

Substituting the value of EI, we get

$$\delta_c = \frac{202.7\ \text{kN·m}^3}{256.0\ \text{MN·m}^2} = 0.792\ \text{mm}$$

This deflection is upward, as shown in the figure.

(c) The maximum downward deflection δ_{max} occurs in span AB at a point E to be located. Let us assume that this point is between D and B. (If it is not, the calculations will so indicate, and then we can begin again by assuming that E is between A and D.) At point E, the deflection curve has a horizontal tangent;

therefore, the area of the M/EI diagram between A and E must equal the angle of rotation θ_a. Denoting the distance from A to E by x_1, we can write the following equation (see the last part of Fig. 7-13):

$$EI\theta_a = \frac{1}{2}(x_1)(20\ \text{kN})(x_1) - \frac{1}{2}(x_1 - 4\ \text{m})(40\ \text{kN})(x_1 - 4\ \text{m})$$

$$= x_1^2(-10\ \text{kN}) + x_1(160\ \text{kN·m}) - 320\ \text{kN·m}^2$$

in which x_1 has units of meters. Substituting $EI\theta_a = 189.3\ \text{kN·m}^2$ into this equation, we get the following quadratic equation for x_1:

$$x_1^2 - 16x_1 + 50.93 = 0$$

Solving by the quadratic formula, we get $x_1 = 4.385$ m (the other root has no physical meaning in this problem). The position of point E between D and B is now determined.

The maximum deflection δ_{max} is numerically equal to the offset of point A from the horizontal tangent at E. Therefore, we can calculate δ_{max} by taking the first moment of the area between A and E with respect to A (see the last part of Fig. 7-13):

$$EI\delta_{max} = \frac{1}{2}(x_1)(20\ \text{kN})(x_1)\left(\frac{2x_1}{3}\right)$$

$$-\frac{1}{2}(x_1 - 4\ \text{m})(40\ \text{kN})(x_1 - 4\ \text{m})\left[4\ \text{m} + \frac{2}{3}(x_1 - 4\ \text{m})\right]$$

Substituting $x_1 = 4.385$ m into the above expressions, we get

$$EI\delta_{max} = 562.2\ \text{kN·m}^3 - 12.63\ \text{kN·m}^3 = 549.6\ \text{kN·m}^3$$

Finally, we calculate δ_{max} in numerical terms:

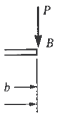
$$\delta_{max} = \frac{549.6\ \text{kN·m}^3}{256.0\ \text{MN·m}^2} = 2.15\ \text{mm}$$

Thus, the maximum downward deflection of the beam has been determined.

In this example, we relied on the geometry of the deflection curve to obtain the desired relationships between angles of rotation and deflections. Such a common-sense procedure often is more efficient than using the proper sign conventions associated with the moment-area theorems.

7.6 METHOD OF SUPERPOSITION

The differential equations of the deflection curve of a beam (Eqs. 7-10) are linear differential equations; that is, all terms containing the deflection v and its derivatives are raised to the first power only. Therefore, solutions of the equations for various loading conditions may be superimposed. Thus, the deflection of the beam caused by several different loads acting simultaneously can be found by superimposing the deflections caused by the loads acting separately. For instance, if v_1 rep-



im with two

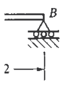
resents the deflection due to a load q_1 and if v_2 represents the deflection due to a load q_2, the total deflection produced by q_1 and q_2 acting simultaneously is $v_1 + v_2$.

To illustrate this idea, consider the cantilever beam shown in Fig. 7-14. This beam supports a uniform load of intensity q over part of the span and a concentrated load P acting at the end. Assume that we want to find the deflection δ_b at the free end. When the load P acts alone, the deflection at B is $PL^3/3EI$, as shown in Example 1 of the preceding section (Eq. 7-39). Also, due to the uniform load acting alone, the deflection is $qa^3(4L - a)/24EI$, as obtained in Example 2 of the preceding section (Eq. 7-41). Hence, the deflection δ_b due to the combined loading is

$$\delta_b = \frac{PL^3}{3EI} + \frac{qa^3(4L - a)}{24EI} \qquad (7\text{-}49)$$

The deflection and angle of rotation at any point of the beam can be found by this procedure.

The method of superposition is most useful when the loading system on the beam can be subdivided into loading conditions that produce deflections that are already known, as illustrated in the example just given. For convenient use in cases of this kind, **tables of beam deflections** are given in Appendix G. Using these tables and the method of superposition, we can find deflections and angles of rotation for many different loading conditions for beams. Some additional examples of this type are given at the end of this section.



with a

Superposition may also be used for distributed loadings by considering an element of the distributed load as if it were a concentrated load and then integrating throughout the region of the load. This procedure can be easily understood from the example shown in Fig. 7-15. The load on the simple beam AB is triangularly distributed over the left-hand half of the beam, and we will assume that the deflection δ at the midpoint is to be found. An element $q\,dx$ of the distributed load can be visualized as a concentrated load. The deflection at the midpoint produced by a concentrated load P acting at distance x from the left end is

$$\frac{Px}{48EI}(3L^2 - 4x^2)$$

which is obtained from Case 5 of Table G-2 in Appendix G. Substituting $q\,dx$ for P in this expression, and noting that $q = 2q_0x/L$, we obtain for the deflection

$$\delta = \int_0^{L/2} \frac{qx\,dx}{48EI}(3L^2 - 4x^2)$$

$$= \frac{q_0}{24LEI}\int_0^{L/2}(3L^2 - 4x^2)x^2\,dx = \frac{q_0L^4}{240EI} \qquad (7\text{-}50)$$

By this same procedure of superimposing elements of the distributed load, we can calculate the angle of rotation θ_a at the left end of the beam. The expression for this angle due to a concentrated load P (see Case 5 of Table G-2) is

$$\frac{Pab(L + b)}{6LEI}$$

In this expression, we must replace P with $2q_0x\,dx/L$, a with x, and b with $L - x$; thus;

$$\theta_a = \int_0^{L/2} \frac{q_0x\,dx}{3L^2EI}(x)(L - x)(2L - x) = \frac{41q_0L^3}{2880EI} \qquad (7\text{-}51)$$

Another illustration of this technique is given in Example 2.

In each of the preceding illustrations, we have used the **principle of superposition** to obtain deflections of beams. This concept is widely used in mechanics and is valid whenever the quantity to be determined is a linear function of the applied loads. Under such conditions, the desired quantity may be found due to each load acting separately, and then the results may be superimposed to obtain the total value due to all loads acting simultaneously. In the case of deflections of beams, the principle of superposition is valid if Hooke's law holds for the material and if the deflections and rotations of the beam are small. The requirement of small rotations ensures that the differential equation of the deflection curve is linear, and the requirement of small deflections ensures that the lines of action of the loads and reactions are not changed significantly from their original positions.

The following examples further illustrate the use of the principle of superposition for calculating deflections of beams.

Example 1

A simple beam AB is acted upon by couples M_0 and $2M_0$ at the ends (see Fig. 7-16). Obtain expressions for the angles of rotation θ_a and θ_b at the ends of the beam and the deflection δ at the middle.

Using Case 7 of Table G-2, we obtain by superposition

$$\theta_a = \frac{M_0L}{3EI} + \frac{(2M_0)L}{6EI} = \frac{2M_0L}{3EI}$$

$$\theta_b = \frac{M_0L}{6EI} + \frac{(2M_0)L}{3EI} = \frac{5M_0L}{6EI}$$

$$\delta = \frac{M_0L^2}{16EI} + \frac{(2M_0)L^2}{16EI} = \frac{3M_0L^2}{16EI}$$

Thus, the required quantities have been found.

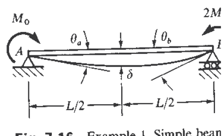
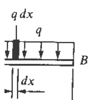


Fig. 7-16 Example 1. Simple bean with couples acting at the ends

Example 2



ple 2. Cantilever rm load over one-

A cantilever beam AB carries a uniform load of intensity q over the right-hand half of its length, as shown in Fig. 7-17. Find the deflection δ_b and the angle of rotation θ_b at the free end.

We begin by considering an element $q\,dx$ of the load located at distance x from the support. This element of load produces a deflection $d\delta$ and an angle $d\theta$ at the free end equal to

$$d\delta = \frac{(q\,dx)(x^2)(3L - x)}{6EI} \qquad d\theta = \frac{(q\,dx)(x^2)}{2EI}$$

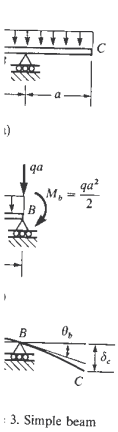
as found from Case 5 of Table G-1. Hence, by integrating, we get

$$\delta_b = \frac{q}{6EI}\int_{L/2}^{L} x^2(3L - x)dx = \frac{41qL^4}{384EI} \qquad (7\text{-}52)$$

$$\theta_b = \frac{q}{2EI}\int_{L/2}^{L} x^2\,dx = \frac{7qL^3}{48EI} \qquad (7\text{-}53)$$

These same results can be obtained more simply by using the formulas in Case 3 of Table G-1 and substituting $a = b = L/2$.

Example 3



: 3. Simple beam

A simple beam with an overhang is loaded as shown in Fig. 7-18a. Find the deflection δ_c at the end of the overhang.

The deflection of point C is made up of two parts: (1) a deflection δ_1 caused by the rotation of the beam axis at support B and (2) a deflection δ_2 caused by the bending of part BC acting as a cantilever beam. To obtain the first part of the deflection, we observe that portion AB of the beam is in the same condition as a simple beam carrying a uniform load and subjected to a couple M_b (equal to $qa^2/2$) and a vertical load (equal to qa) acting at the right-hand end, as shown in Fig. 7-18b. The angle θ_b at end B (see Cases 1 and 7 of Table G-2) is:

$$\theta_b = -\frac{qL^3}{24EI} + \frac{M_bL}{3EI} = \frac{qL(4a^2 - L^2)}{24EI}$$

in which clockwise rotation is positive. The deflection δ_1 of point C, due to the rotation at B, is equal to $a\theta_b$, or

$$\delta_1 = \frac{qaL(4a^2 - L^2)}{24EI}$$

This deflection is positive when downward.

The bending of the overhang itself produces a downward deflection δ_2 at C. This deflection is equal to the deflection of a cantilever beam of length a (see Case 1 of Table G-1):

$$\delta_2 = \frac{qa^4}{8EI}$$

The deflection of point C, assumed to be positive when downward, is

$$\delta_c = \delta_1 + \delta_2 = \frac{qa}{24EI}(3a^3 + 4a^2L - L^3) \qquad (7\text{-}54)$$

From this result, we can show that, when a is less than $L(\sqrt{13} - 1)/6$, or 0.434 L, the deflection δ_c is negative and point C deflects upward.

The shape of the deflection curve for the beam in this example is shown in Fig. 7-18c for the case where a is large enough ($a > 0.434L$) to produce a downward deflection at C and small enough ($a < L$) to ensure that the reaction at A is upward. Under these conditions the beam has a positive bending moment from A to a point such as D; hence, the deflection curve is convex downward in this part of the beam. From D to C, the bending moment is negative, and the deflection curve is convex upward. Point D, at which the curvature of the axis of the beam is zero (because the bending moment is zero), is a point of inflection. The curvature of the deflection curve changes sign at this point.

Example 4

Determine the deflection δ_b at the hinge B for the compound beam shown in Fig. 7-19. Note that the beam is composed of two parts: (1) a beam AB, simply supported at A, and (2) a cantilever beam BC, fixed at C. The two beams are linked together by a pin connection at B.

Considering beam AB as a free body, we see that it has vertical reactions $P/3$ and $2P/3$ at ends A and B, respectively. Therefore, beam BC is in the condition of a cantilever beam subjected to a uniform load of intensity q and a concentrated load at the end equal to $2P/3$. The deflection of the end of this cantilever, which is the same as the deflection of the hinge, is

$$\delta_b = \frac{qb^4}{8EI} + \frac{2Pb^3}{9EI}$$

as found from Cases 1 and 4 of Table G-1.

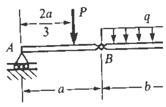


Fig. 7-19 Example 4. Co beam with a hinge

7.7 NONPRISMATIC BEAMS

The methods presented in the preceding sections for calculating deflections of prismatic beams (that is, beams with constant cross sections throughout their lengths) can also be used to find deflections of nonprismatic beams. Such beams include those with different cross-sectional areas in various parts of the beam (see Fig. 7-20 for an example) and tapered beams (see Fig. 7-21). When a beam has abrupt changes in cross-sectional dimensions, there are local stress concentrations at the points where changes occur; however, these local stresses have no noticeable effect on the calculation of deflections. For a tapered beam, the bending theory derived previously for a prismatic beam gives satisfactory results provided that the angle of taper is small.